

Superconductors with topological order

M.C. Diamantini^{1,a}, P. Sodano^{1,2,b}, and C.A. Trugenberger^{3,c}

¹ INFN and Dipartimento di Fisica, University of Perugia, via A. Pascoli, 06100 Perugia, Italy

² Progetto Lagrange, Fondazione C.R.T. e Fondazione I.S.I. c/o Dipartimento di Fisica, Politecnico di Torino, Corso Duca degli Abruzzi 24, Torino, 10124, Italy

³ InfoCodex S.A., Av. Louis-Casai 18, 1209 Geneva, Switzerland

Received 12 July 2006

Published online 7 September 2006 – © EDP Sciences, Società Italiana di Fisica, Springer-Verlag 2006

Abstract. We propose a mechanism of superconductivity in which the order of the ground state does not arise from the usual Landau mechanism of spontaneous symmetry breaking but is rather of topological origin. The low-energy effective theory is formulated in terms of emerging gauge fields rather than a local order parameter and the ground state is degenerate on topologically non-trivial manifolds. The simplest example of this mechanism of superconductivity is concretely realized as global superconductivity in Josephson junction arrays.

PACS. 11.10.-z Field theory – 74.20.Mn Nonconventional mechanisms (spin fluctuations, polarons and bipolarons, resonating valence bond model, anyon mechanism, marginal Fermi liquid, Luttinger liquid, etc.)

1 Introduction

The discovery of the fractional quantum Hall [1] effect has revealed the existence of a new state of matter characterized by a new type of order: topological order [2]. Topological order is a particular type of quantum order describing zero-temperature properties of a ground state with a gap for all excitations. Its hallmarks are the degeneracy of the ground state on manifolds with non-trivial topology, and excitations with fractional spin and statistics, called anyons [3]. The long-distance properties of these topological fluids are described by Chern-Simons field theories [4] with compact gauge group, which break P- and T-invariance. Other examples of P- and T-breaking topological fluids are given by chiral spin liquids [5].

After Laughlin's discovery of topological quantum fluids, it was conjectured that a similar mechanism, based on anyon condensation, could be at the origin of high- T_c superconductivity [3]. Unfortunately, there is no evidence of the associated broken P- and T-invariance in the high- T_c materials.

Here we propose a superconductivity mechanism which is based on a topologically ordered ground state rather than on the usual Landau mechanism of spontaneous symmetry breaking. Contrary to anyon superconductivity it works in any dimension and it preserves P- and

T-invariance. In particular we will discuss the low-energy effective field theory, what would be the Landau-Ginzburg formulation for conventional superconductors.

Topologically ordered superconductors have a long-distance hydrodynamic action which can be entirely formulated in terms of generalized compact gauge fields, the dominant term being the topological BF action.

BF theories are topological theories that can be defined on manifolds M_{d+1} of any dimension (here d is the number of spatial dimensions) and play a crucial role in models of two-dimensional gravity [6]. In [7] we have shown that the BF term also plays a crucial role in the physics of Josephson junction arrays.

The BF term [8] is the wedge product of a p -form B and the curvature dA of a $(d-p)$ form A :

$$S_{BF} = \frac{k}{2\pi} \int_{M_{d+1}} B_p \wedge dA_{d-p},$$

where k is a dimensionless coupling constant. This can also be written as

$$S_{BF} = \frac{k}{2\pi} \int_{M_{d+1}} A_{d-p} \wedge dB_p. \quad (1)$$

The integration by parts does not imply any surface term since we will concentrate on compact spatial manifolds without boundaries and we require that the fields go to pure gauge configuration at infinity in the time direction. Indeed this action has a generalized Abelian gauge

^a e-mail: cristina.diamantini@pg.infn.it

^b e-mail: pasquale.sodano@pg.infn.it

^c e-mail: ca.trugenberger@InfoCodex.com

symmetry under the transformation

$$B \rightarrow B + \eta,$$

where η is a closed p form: $d\eta = 0$. Gauge transformations:

$$A \rightarrow A + \xi,$$

with ξ a closed $(d-p)$ form instead, change the action by a surface term. This, however vanishes with the boundary conditions we have chosen.

Here we will be interested in the special case where A_1 is a 1-form and, correspondingly, B_{d-1} is a $(d-1)$ -form:

$$S_{BF} = \frac{k}{2\pi} \int_{M_{d+1}} A_1 \wedge dB_{d-1}. \quad (2)$$

In the special case of $(3+1)$ dimensions, B is the well-known Kalb-Ramond tensor field $B_{\mu\nu}$ [9].

In the application to superconductivity, the conserved current $j_1 = *dB_{d-1}$ represents the charge fluctuations, while the generalized current $j_{d-1} = *dA_1$ describes the conserved fluctuations of $(d-2)$ -dimensional vortex lines. As a consequence, the form B_{d-1} must be considered as a pseudo-tensor, while A_1 is a vector, as usual. The BF coupling is thus P- and T-invariant.

The low-energy effective theory of the superconductor can be entirely expressed in terms of the generalized gauge fields A_1 and B_{d-1} . The dominant term at long distances is the BF term; the next terms in the derivative expansion of the effective theory are the kinetic terms for the two gauge fields (for simplicity of presentation we shall assume relativistic invariance), giving:

$$S_{TM} = \int_{M_{d+1}} \frac{-1}{2e^2} dA_1 \wedge *dA_1 + \frac{k}{2\pi} A_1 \wedge dB_{d-1} + \frac{(-1)^{d-1}}{2g^2} dB_{d-1} \wedge *dB_{d-1}, \quad (3)$$

where e^2 and g^2 are coupling constants of dimension m^{-d+3} and m^{d-1} respectively.

The BF-term is the generalization to any number of dimensions of the Chern-Simons mechanism for the topological mass [10]. To see this let us now compute the equation of motion for the two forms A and B :

$$\frac{1}{g^2} d * dB_{d-1} = \frac{k}{2\pi} dA_1, \quad (4)$$

and

$$\frac{1}{e^2} d * dA_1 = \frac{k}{2\pi} dB_{d-1}. \quad (5)$$

Applying $d*$ on both sides of (4) and (5) we obtain

$$\begin{aligned} d * d * dA_1 - \frac{ke^2}{2\pi} d * dB_{d-1} &= 0, \\ d * d * dB_{d-1} - \frac{kg^2}{2\pi} d * dA_1 &= 0. \end{aligned} \quad (6)$$

The expression $*d*$ is proportional to δ , the adjoint of the exterior derivative [11]. Substituting $d * dB_{d-1}$ and

$d * dA_1$ in (6) with the expression coming from (4) and (5) we obtain

$$\begin{aligned} (\Delta + m^2) dA_1 &= 0, \\ (\Delta + m^2) dB_{d-1} &= 0, \end{aligned} \quad (7)$$

where $\Delta = d\delta$ (when acting on an exact form) and $m = \frac{keg}{2\pi}$ is the topological mass. This topological mass plays the role of the gap characterizing the superconducting ground state. Note that the gap arises here from a topological mechanism and not from a local order parameter acquiring a vacuum expectation value. Equations (4) and (5) tell us that charges are sources for vortex line currents encircling them and vice versa. This is the coupling between charges and vortices at the origin of the gap. A related mechanism for topological mass generation in $(3+1)$ -dimensional gauge theories is the generalization of the Schwinger mechanism proposed in [12].

Let us now consider the special case of $(2+1)$ dimensions ($d = 2$). In this case also B becomes a (pseudo-vector) 1-form and, correspondingly the BF term reduces to a mixed Chern-Simons term. This can be diagonalized by a transformation $A = \frac{a+b}{2}$, $B = a - b$, giving

$$S_{BF}(d = 2) = \frac{k}{4\pi} \int a \wedge da - \frac{k}{4\pi} \int b \wedge db. \quad (8)$$

The result is a doubled Chern-Simons model for gauge fields of opposite chirality. This action, including its non-Abelian generalization with kinetic terms was first considered in [13]. It is the simplest example of the class of P- and T-invariant topological phases of strongly correlated $(2+1)$ -dimensional electron systems considered in [14]. Indeed, the BF term is the natural generalization of such doubled Chern-Simons models to any dimension. Doubled (or mixed) Chern-Simons models are thus particular examples in two spatial dimensions of a wider class of P- and T-invariant topological fluids that have a superconducting phase. These fluids are described by the topological BF theory with compact support for both gauge fields.

Topological BF models provide also a generalization of anyons to arbitrary dimensions. While in $(2+1)$ dimensions fractional statistics arises from the representations of the braid group, encoding the exchange of particles, in $(3+1)$ dimensions it arises from the adiabatic transport of particles around vortex strings and, in $(d+1)$ dimensions, from the motion of an hypersurface Σ_h around another hypersurface Σ_{d-h} . The relevant group in this case is the motion group and the statistical parameter is given by $\frac{2\pi}{k}h(d-h)$, where k is the BF coupling constant [15].

Let us now illustrate the mechanism of superconductivity. To this end we shall from now on consider only rational $k = \frac{k_1}{k_2}$ with k_i integers, and specialize to manifolds $M_{d+1} = M_d \times R_1$, with R_1 representing the time direction.

The compactness of the gauge fields allows for the presence of topological defects, both electric and magnetic. The electric topological defects couple to the form A_1 and

are string-like objects described by a singular closed 1-form Q_1 . Magnetic topological defects couple to the form B_{d-1} and are closed $(d-1)$ -branes described by a singular $(d-1)$ -dimensional form Ω_{d-1} . These forms represent the singular parts of the field strengths dA_1 and dB_{d-1} , allowed by the compactness of the gauge symmetries [16], and are such that the integral of their Hodge dual over any hypersurface of dimensions d and 2, respectively, is 2π times an integer as can be easily derived using a lattice regularization. Contrary to the currents j_1 and j_{d-1} , which represent charge- and vortex-density waves, the topological defects describe localized charges and vortices. In the effective theory these have structure on the scale of the ultraviolet cutoff.

We will not discuss here the conditions for the condensation of topological defects, but we will show, instead that the phase of electric condensation describes a superconducting phase in any dimension. A detailed analysis would require the use of an ultraviolet regularization. Here we will present a formal derivation implying the ultraviolet regularization. A detailed derivation on the lattice will follow in a forthcoming publication [17].

In the phase in which electric topological defects condense (while magnetic ones are dilute) the partition function requires a formal sum also over the form Q_1

$$Z = \int \mathcal{D}A \mathcal{D}B \mathcal{D}Q \times \exp \left[i \frac{k}{2\pi} \int_{M_{d+1}} (A_1 \wedge dB_{d-1} + A_1 \wedge *Q_1) \right]. \quad (9)$$

Let us now compute the expectation value of the 't Hooft operator, $\langle L_H \rangle$, which represents the amplitude for creating and separating a pair of vortices with fluxes $\pm\phi$:

$$\begin{aligned} \langle L_H \rangle &= \frac{1}{Z} \int \mathcal{D}A \mathcal{D}B \mathcal{D}Q \\ &\times \exp \left[i \frac{k}{2\pi} \int_{M_{d+1}} (A_1 \wedge dB_{d-1} + A_1 \wedge *Q_1) \right. \\ &\quad \left. + i \frac{k}{2\pi} \phi \int_{S_{d-1}} B_{d-1} \right]. \end{aligned} \quad (10)$$

Using Stokes' theorem we can rewrite

$$\int_{S_{d-1}} B_{d-1} = \int_{S_d} dB_{d-1}, \quad (11)$$

where the surface S_d is such that $\partial S_d \equiv S_{d-1}$ and represents a compact orientable surface on M_d . Inserting (11) in (10) and integrating over the field A we obtain:

$$\begin{aligned} \langle L_H \rangle &\propto \int \mathcal{D}B \mathcal{D}Q \delta(dB_{d-1} + *Q_1) \\ &\times \exp \left[i \frac{k}{2\pi} \phi \int_{S_d} dB_{d-1} \right]. \end{aligned} \quad (12)$$

Integrating over B gives then:

$$\langle L_H \rangle \propto \int \mathcal{D}Q \exp \left[-i \frac{k}{2\pi} \phi \int_{S_d} *Q_1 \right]. \quad (13)$$

The Poisson summation formula implies finally that the 't Hooft loop expectation value vanishes for all flux strengths ϕ different from

$$\frac{\phi}{k_2} = \frac{2\pi}{k_1} n \quad n \in N. \quad (14)$$

This is nothing else than the Meissner effect, illustrating that the electric condensation phase is superconducting. Indeed, the electric condensate carries k_1 fundamental charges of unit $1/k_2$ as is evident from (9), and correspondingly vortices must carry an integer multiple of the fundamental fluxon $2\pi/(k_1/k_2)$. All other vorticities are confined: in this purely topological long-distance theory the confining force is infinite; including the higher order kinetic terms (3) and the UV cutoff one would recover a generalized area law.

Another way to see this is to compute the current induced by an external electromagnetic field A_{ext} . The corresponding coupling is $\int_{M_{d+1}} A_{\text{ext}} \wedge (*j_1 + *Q_1) \propto \int_{M_{d+1}} A_{\text{ext}} \wedge (dB_{d-1} + *Q_1)$. Since A_{ext} can be entirely reabsorbed in a redefinition of the gauge field A_1 , the induced current vanishes identically, $j_{\text{ind}} = 0$. This is just the London equation in the limit of zero penetration depth. Including the higher-order kinetic terms for the gauge fields and the UV cutoff one would again recover the standard form of the London equation.

Associated with the confinement of vortices there is a breakdown of the original $U(1)$ matter symmetry under transformations $A_1 \rightarrow A_1 + d\lambda$. To see this let us consider the effect of such a transformation on the partition function (9) with an electric condensate. Upon integration by parts, the exponential of the action acquires a multiplicative factor

$$\exp i \frac{k_1}{2\pi k_2} \left(\int_{M_d, t=+\infty} \lambda \wedge *Q_1 - \int_{M_d, t=-\infty} \lambda \wedge *Q_1 \right). \quad (15)$$

Assuming a constant λ , we see that the only values for which the partition function remains invariant are

$$\lambda = 2\pi n \frac{k_2}{k_1}, \quad n = 1 \dots k_1, \quad (16)$$

which shows that the global symmetry is broken from $U(1)$ to Z_{k_1} . Note that this is not the usual Landau mechanism of spontaneous symmetry breaking. Indeed, there is no local order parameter and the order is characterized rather by the expectation value of non-local, topological operators.

The hallmark of topological order is the degeneracy of the ground state on manifolds with non-trivial topology as shown by Wen [2]. In (2+1) dimensions the degeneracy for the mixed Chern-Simons term was proven in [18] for the case of integer coefficient k of the Chern-Simons term.

The degeneracy of the ground state of the BF theory on a manifold with non-trivial topology was proven in [19] in (3+1) dimensions. This result can be generalized to compact topological BF models in any number of dimensions [15]. Consider the model (1) with $k = \frac{k_1}{k_2}$ on a manifold $M_d \times R_1$, with M_d a compact, path-connected, orientable d -dimensional manifold without boundaries. The

degeneracy of the ground state is expressed in terms of the intersection matrix M_{mn} [20] with $m, n = 1 \dots N_p$ and N_p the rank of the matrix, between p -cycles and $(d-p)$ -cycles. N_p corresponds to the number of generators of the two homology groups $H_p(M_d)$ and $H_{d-p}(M_d)$ and is essentially the number of non-trivial cycles on the manifold M_d . The degeneracy of the ground state is given by $|k_1 k_2 M|^{N_p}$, where M is the integer-valued determinant of the linking matrix. In our case $p = (d - 1)$ and the degeneracy reduces to

$$|k_1 k_2 M|^{N_{d-1}}. \quad (17)$$

In this paper we have derived a superconductivity mechanism which is not based on the usual Landau theory of spontaneous symmetry breaking. Our considerations here focused on the low-energy effective theory in order to expose the physical basis of the topological superconductivity mechanism. It is however crucial to stress that the simplest example ($k = 1$) of this type of topological superconductivity is concretely realized as the global superconductivity mechanism in planar Josephson junction arrays, as we have shown in [7]. Naturally, it would be most interesting to find examples of devices or materials realizing this superconductivity mechanism with more complex degeneracy patterns. Planar Josephson junction arrays with pertinent charge and magnetic frustration seem to be an interesting candidate [21] to realize topological superconductivity with rational k ; furthermore, the existence of such non-conventional superconductors is also supported by purely algebraic considerations [22].

References

1. For a review see: R.E. Prange, S.M. Girvin, *The Quantum Hall Effect* (Springer-Verlag, New York, 1990)
2. For a review see: X.-G. Wen, *Advances in Physics* **44**, 405 (1995), e-print [arXiv:cond-mat/9506066](https://arxiv.org/abs/cond-mat/9506066)
3. For a review see: F. Wilczek, *Fractional Statistics and Anyon Superconductivity* (World Scientific, Singapore, 1990)
4. X.-G. Wen, A. Zee, *Nucl. Phys. B* **15**, 135 (1990)
5. V. Kalmeyer, R.B. Laughlin, *Phys. Rev. Lett.* **59**, 2095 (1987); X.-G. Wen, F. Wilczek, A. Zee, *Phys. Rev. B* **39**, 11413 (1989)
6. K. Isler, C.A. Trugenberger, *Phys. Rev. Lett.* **63**, 834 (1989); A.H. Chamseddine, A. Wyler, *Phys. Lett. B* **228**, 75 (1989)
7. M.C. Diamantini, P. Sodano, C.A. Trugenberger, *Nucl. Phys. B* **474**, [FS] 641 (1996)
8. For a review see: D. Birmingham, M. Blau, M. Rakowski, G. Thompson, *Phys. Rep.* **209**, 129 (1991)
9. M. Kalb, P. Ramond, *Phys. Rev. D* **9**, 2273 (1974)
10. R. Jackiw, S. Templeton, *Phys. Rev. D* **23**, 2291 (1981); J. Schonfeld, *Nucl. Phys. B* **185**, 157 (1981); S. Deser, R. Jackiw, S. Templeton, *Phys. Rev. Lett.* **48**, 975 (1982); *Ann. Phys. (N.Y.)* **140**, 372 (1982)
11. T. Eguchi, P.B. Gilkey, A.J. Hanson, *Phys. Rep.* **66**, 213 (1980)
12. G. Dvali, R. Jackiw, S.Y. Pi, *Topological Mass Generation in Four Dimensions*, e-print [arXiv:hep-th/0511175](https://arxiv.org/abs/hep-th/0511175)
13. R. Jackiw, S.Y. Pi, *Phys. Lett. B* **403**, 297 (1997)
14. M. Freedman, C. Nayak, K. Shtengel, K. Walker, Z. Wang, *Ann. Phys.* **310**, 428 (2004)
15. R.J. Szabo, *Ann. Phys.* **280**, 163 (2000)
16. For a review see: A.M. Polyakov, *Gauge Fields and String* (Harwood Academic Publisher, 1987) Chur
17. M.C. Diamantini, P. Sodano, C.A. Trugenberger, in preparation
18. D. Wesolowski, Y. Hosotani, C.-L. Ho, *Int. J. Mod. Phys. A* **9**, 969 (1994)
19. M. Bergeron, G.W. Semenoff, R.J. Szabo, *Nucl. Phys. B* **437**, 695 (1995)
20. R. Bott, L.W. Tu, *Differential Forms in Algebraic Topology* (Springer-Verlag, New York, 1986)
21. M.C. Diamantini, P. Sodano, *Phys. Rev. B* **45**, 5737 (1992)
22. C.A. Trugenberger, *Nucl. Phys. B* **716**, 509 (2005)